Table 1 Variation in α_{crit}

Number of terms	α_{crit} (to 2 significant figures)
2	5.5
5	5.6
10	5.7
15	5.8

lem originating from a second-order linear differential equation having homogeneous boundary conditions. Theoretical treatment indicates that a condition for the eigenvalues of such a system to be always real is that the quotient of the coefficients of the second derivative and the "λy" term be positive in the interval.[4] For Eq. (2) the coefficients are constants and the result follows trivially.

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Normal Shock Location of Underexpanded Gas-Particle Jets

ANDREW B. BAUER*
Aeronutronic, Newport Beach, Calif.

Nomenclature

η = $r/R(x)$
x = axial distance from the nozzle exit plane
r = radial distance from axis
$R(x)$ = radius of the outer edge of the gas cloud or plume
R_0 = radius of the nozzle exit plane
v = velocity normal to the axis
ρ = gas density
ρ_0 = averaged value of ρ at $x = 0$
a_0 = sonic speed at $x = 0$
$B(\gamma)$ = $1/(\gamma - 1)$
$D(\gamma)$ = an unknown function

Introduction

AN important consideration in underexpanded rocket nozzle plumes is the location of the normal shock, since the shock structure affects the radiation of the plume. This note makes use of a simple physical model to show how solid particles affect the normal shock location.

The model is based on the assumption that the ambient pressure is much smaller than the jet exit pressure, producing a greatly underexpanded jet. Under these conditions, the effect of the ambient pressure on the flow near the plume axis may be neglected. The flow parameters near the axis are calculated by the analogy between the unsteady expansion of a cylindrical gas cloud and the steady hypersonic flow from a nozzle. Hence, the model applies only to nozzles that have a sufficiently high exit Mach number.

The model also uses the assumption of no particle lag in the supersonic portion of the plume, although it treats both no-lag and complete-lag flows behind the normal shock. Except for the recognition of particle drag, viscous effects are neglected throughout the analysis.

The normal shock location is based on the method given by Adamson[1] and by Adamson and Nicholls,[2] since it is simpler to apply than the methods of D'Attorre and Harshbarger[3] and Eastman and Radtke.[4]

Expansion of the Rocket Plume

The hypersonic blast or expanding gas-cloud analogy used here has been pointed out both by Mirels and Mullen[5] and by Greifinger and Cole.[6] The essential point is that the flow problem parallel to the rocket plume axis may be decoupled mathematically from the flow normal to the axis by neglecting terms of order δ^2 as compared with 1, where δ is a characteristic slope of the plume streamlines. This will be true if the nozzle-exit plane Mach number M_0 is such that $M_0^2 \gg 1$, provided that the gas specific heat ratio γ is not nearly equal to 1. Then, the radial flow problem is just that of the gas cloud.

For the expansion of a cylindrical gas cloud, both Sedov[7] and Keller[8] show that, after a long time, a self-similar solution exists in the form of

$$\rho/\rho_0 = D(\gamma)(R_0M_0/x)^2(1 - \eta^2)^{B(\gamma)} \quad (1)$$

$$v/a_0 = \eta(R_0M_0/x)(R/R_0) \quad (2)$$

where the dimensionless time t has been replaced by the dowhstream distance (x/R_0M_0) and where R is given for large values of $x/(M_0R_0)$ as

$$R(x) = [2^{1/2}/(\gamma - 1)](x/M_0R_0)R_1 \quad (3)$$

These equations are valid for $R^{2(\gamma-1)}$ large compared to $R_0^{2(\gamma-1)}$; R_1 will be defined later.

At $x = 0$, the rocket nozzle has a density distribution that is, in general, different from $(1 - \eta^2)^{1/(\gamma-1)}$. Hence, the question remains of whether the distribution will tend to approach the form $(1 - \eta^2)^{1/(\gamma-1)}$ for large values of (x/M_0R_0). Mirels and Mullen[5] settle the question by equating the mass-flow rate past $x = 0$ to the mass-flow rate at large values of (x/M_0R_0), and by doing the same for energy-flow rate. They also assume the condition that $R_1 = R_0$, where R_1 is the radius of the self-similar density distribution at $x = 0$ that is equivalent to the real rocket nozzle density distribution at $x = 0$. These three conditions determine the three unknowns $D(\gamma)$, $B(\gamma)$, and R_1. This leads to $B(\gamma) = 2/(\gamma - 1)$, although $1/(\gamma - 1)$ is the only self-similar exponent that satisfies the equations of motion.[7, 8] Of course, it has not been shown that the plume would be stable in or tend toward the self-similar form, but in the absence of any evidence to the contrary, it will be assumed that the plume approaches this form, so that $B(\gamma) = 1/(\gamma - 1)$. Then we may use the mass and energy conservation equations to calculate $D(\gamma)$ and the ratio (R_1/R_0).

Mass conservation gives

$$\rho_0 u_0 \pi R_0^2 = R^2 \int_0^1 \rho u_m 2\pi\eta dy \quad (4)$$

and energy conservation gives

$$\left(\frac{\rho_0 a_0^2}{\gamma - 1}\right)\pi R_0^2 = R^2 \int_0^1 \rho k^2 \frac{v^2}{2} 2\pi\eta dy \quad (5)$$

where u_0 = axial velocity at $x = 0$, u_m = axial velocity at large x, and $k = u_m/u_0$. The distinction between u_0 and u_m is added for the purpose of extending the results to cases where M_0 is not strictly hypersonic, that is, to cases where the gas experiences a small but significant axial acceleration just downstream of the exit plane. This, of course, implies some coupling between the axial and the radial-flow problems, which is neglected here. In Eq. (4), the mass flux at $x = 0$ is equated to the mass flux at large values of x/R_0M_0.

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* Principal Scientist, Fluid Mechanics Department. Member AIAA.

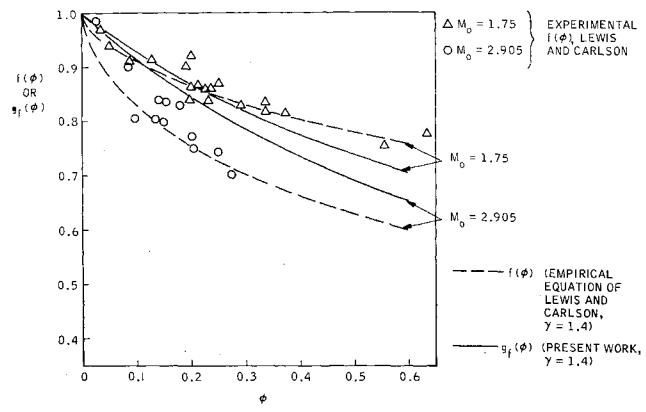

Fig. 1 Normalized shock location distance vs particle mass-flow ratio.

In Eq. (5), the internal energy flux $x = 0$ is equated to the radial-kinetic energy flux at large values of x/R_0M_0, since the internal energy $\to 0$ as $(x/R_0M_0) \to \infty$. These calculations are closely analogous to Eqs. (27a) and (27b) given by Mirels and Mullen.[5] Insertion of Eqs. (1–3) into Eqs. (4) and (5) leads to

$$D(\gamma) = \gamma(\gamma - 1)/2(2\gamma - 1) \tag{6}$$

$$R_1/R_0 = [(2\gamma - 1)/k]^{1/2} \tag{7}$$

so that

$$\frac{\rho}{\rho_0} = \frac{\gamma(\gamma - 1)}{2(2\gamma - 1)} \left(\frac{M_0R_0}{x}\right)^2 (1 - \eta^2)^{1/(\gamma-1)} \tag{8}$$

The preceding was calculated for the plume expansion into a vacuum, but it is assumed to be valid for the present purpose as long as the ambient pressure p_∞ is sufficiently small. This is taken to be $p_0/p_\infty \gg 1$, where p_0 is the stream static pressure at $x = 0$. Then the ambient pressure is not great

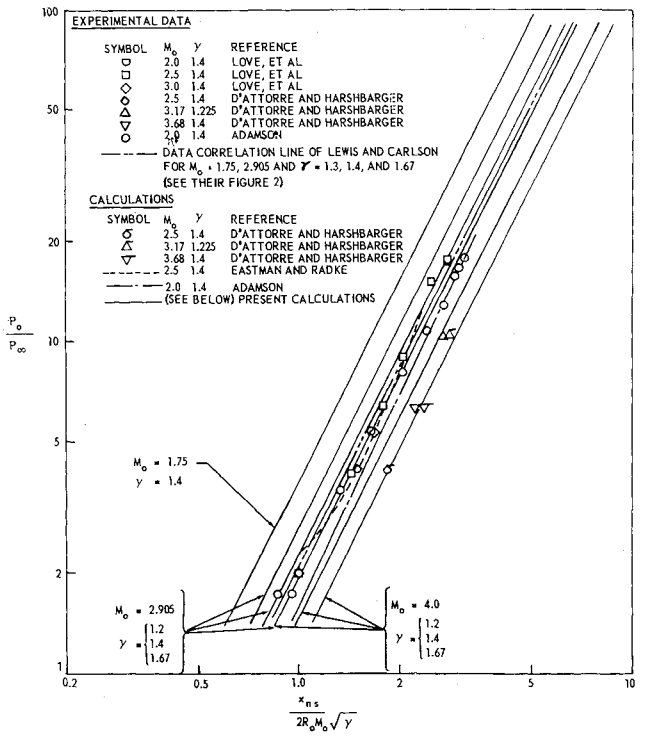

Fig. 2 Ratio of nozzle exit pressure to ambient pressure vs the normal shock-location parameter for zero particle addition.

Table 1 A Comparison of Greifinger's results with the present $D(\gamma)$

γ	$D(\gamma)$	$D_G(\gamma)$
1.11	0.050	0.055
1.2	0.086	0.096
1.4	0.156	0.190
1.67	0.239	0.330

enough to affect the plume density significantly in the region upstream of the normal-shock location.

Greifinger[9] has numerically solved the axisymmetric hypersonic vacuum-plume equations at large distances from the nozzle and has found that the density is actually a little greater than that given previously. The $D(\gamma)$ given previously compares to Greifinger's result $D_G(\gamma)$ as shown in Table 1.

Particle Addition to the Plume

In general, particle addition to high-speed flows greatly complicates the analysis, as discussed by Hoglund.[10] For sufficiently small particle sizes, such that particle lag is negligible, Hoglund shows that the mixture behaves like a gas with an isentropic exponent λ given by

$$\lambda = \gamma\{(1 + \varphi C/C_{p_g})/(1 + \gamma\varphi C/C_{p_g})\} \tag{9}$$

where φ is the ratio of particle mass-flow rate to gas mass-flow rate, and C/C_{p_g} is the ratio of particle specific heat to gas specific heat at constant pressure.

Normal Shock Location

The hypersonic approximation for the pressure behind a normal shock is,[11] for no particle lag,

$$\bar{p}_{ns} = \bar{\rho}\bar{u}_m^2[2/(\lambda + 1)] \tag{10}$$

where barred symbols will denote the presence of particles in the flow; i.e., $\bar{\rho} = \rho(1 + \varphi)$, and $\bar{u}_m$ is now the particle and gas velocity just upstream of the shock. This velocity is related to $\bar{u}_0$ by the isentropic relation[11]

$$\frac{\bar{u}_m^2}{\bar{u}_0^2} = \frac{\{1 + [(\lambda - 1)/2]\bar{M}_0^2\}}{\{(\bar{M}_0^2/\bar{M}_m^2) + [(\lambda - 1)/2]\bar{M}_0^2\}}$$

where $\bar{M}_m$ is the Mach number just ahead of the shock. Because of the previous requirement that $R^{2(\gamma-1)} \gg R_0^{2(\gamma-1)}$, one can show that

$$\bar{p}_m/\bar{p}_0 = D(\lambda)(\bar{M}_0R_0/x_m)^2 \ll 1$$

at $r = \eta = 0$, and, because of the isentropic relation[11]

$$\frac{\bar{p}_m}{\bar{p}_0} = \left\{\frac{1 + [(\lambda - 1)/2]\bar{M}_0^2}{1 + [(\lambda - 1)/2]\bar{M}_m^2}\right\}^{1(\lambda-1)}$$

it is easy to show that

$$\bar{M}_0^2/\bar{M}_m^2 \ll 1$$

which leads to the relation

$$\bar{u}_m/\bar{u}_0 = \{1 + [2/(\lambda - 1)\bar{M}_0^2]\}^{1/2} \tag{11}$$

Then, by using the rule of Adamson and Nicholls[2] that the normal shock is located such that $\bar{p}_{ns} = p_\infty$, and by using Eqs. (8, 10, and 11), one obtains the shock locations $\bar{x}_{ns}$ as

$$\frac{\bar{x}_{ns}}{R_0} = \left[\frac{\lambda^2(\lambda - 1)}{(\lambda + 1)(2\lambda - 1)} \left(\frac{p_0}{p_\infty}\right) \left(1 + \frac{2}{(\lambda - 1)\bar{M}_0^2}\right)\right]^{1/2} \bar{M}_0^2 \tag{12}$$

The function $g(\varphi)$, which is defined as the ratio of $\bar{x}_{ns}$ to

x_{ns}, where x_{ns} is the shock location without particle addition (but with the same nozzle geometry and exit pressure p_0 used in the calculation of $\bar{x}_{ns}$), is given by

$$g(\varphi) = \left[\frac{\lambda^2(\lambda-1)}{(\lambda+1)(2\lambda-1)}\,\frac{(\gamma+1)(2\gamma-1)}{\gamma^2(\gamma-1)}\times\frac{\{1+[2/(\lambda-1)\bar{M}_0^2]\}}{\{1+[2/(\gamma-1)M_0^2]\}}\right]^{1/2}\left(\frac{\bar{M}_0}{\bar{M}_9}\right)^2 \quad (13)$$

In some cases, the particles may be sufficiently large so that the flow behind the normal shock may be considered "frozen"; that is, the particles will fail to follow the flow behind the shock. In this event, Eqs. (8, 10, 12, and 13) are replaced by (8f, 10f, 12f, and 13f):

$$\rho/\bar{\rho}_0 = [D(\lambda)/(1+\varphi)]\bar{M}_0^2(R_0/x)^2 \quad (8f)$$

$$\bar{p}_{nsf} = \rho\bar{u}_m^2(2/\gamma+1) \quad (10f)$$

$$\left(\frac{\bar{x}_{ns}}{R_0}\right)_f = \left[\frac{\lambda^2(\lambda-1)\{1+[2/(\lambda-1)\bar{M}_0^2]\}}{(\gamma+1)(2\lambda-1)(1+\varphi)}\,\frac{p_0}{p_\infty}\right]^{1/2}\bar{M}_0^2 \quad (12f)$$

$$g_f(\varphi) = \left[\frac{\lambda^2(\lambda-1)}{(2\lambda-1)(1+\varphi)}\cdot\frac{(2\gamma-1)}{\gamma^2(\gamma-1)}\times\frac{\{1+[2/(\lambda-1)\bar{M}_0^2]\}}{\{1+[2/(\gamma-1)M_0^2]\}}\right]^{1/2}\left(\frac{\bar{M}_0}{\bar{M}_0}\right)^2 \quad (13f)$$

Results Compared with Experiment

Lewis and Carlson[12] have correlated experimental data showing the change in normal shock location as a function of particle mass-flow ratio φ. Their data are plotted in Fig. 1 in terms of their $f(\varphi)$, which is just the ratio $\bar{x}_{ns}/x_{ns}$ with no qualification as to particle lag conditions. Since it can be argued that the particle lag in these experiments should be important only behind the normal shock, the present $g_f(\varphi)$ is compared to the experimental data in Fig. 1. (The corresponding values of $g(\varphi)$ are larger than $g_f(\varphi)$ but less than 1.)

Since previous equations are applicable to gas-only flows ($\varphi = 0$; $\lambda = \gamma$), Fig. 2 is included to show the comparison between the present calculation and several other calculations and experiments. The data of Love et al. is found in Ref. 13.

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Direct Measurement of Velocity by Hot-Wire Anemometry

ANDREW B. BAUER*

Aeronutronic Applied Research Laboratory, Newport Beach, Calif.

Introduction

THIS note reports a new method for measuring fluid velocity. The method is of special interest since it can be combined with the standard hot-wire measurement techniques[1, 2] to determine the fluid pressure p, density ρ, and temperature T, as well as the three velocity components. This is done with a single, small probe.

Standard hot-wire techniques can be used to measure the total temperature T_0 and the product ρU, where U is the velocity component normal to the wire, and where the velocity component parallel to the wire is small as compared with U. Then additional knowledge of the flow, such as constant density conditions or pitot-static tube measurements, will permit U to be determined. The new method determines U independently, so that ρ is calculated easily. The measurement of T_0 then makes it a simple exercise to calculate T and p.

The technique makes use of two hot wires mounted on a simple probe. The wires are held parallel to each other. The velocity direction is determined by heating wire no. 1 and sensing the heated wake with wire no. 2, as illustrated in Fig. 1. This is accomplished most easily by first rotating the entire probe through an angle φ about an axis parallel to the needle supports. When wire no. 2 is normal to the flow, its temperature and resistance will be a minimum. The resistance is sensed by passing a small constant current through the wire and by measuring the wire voltage. Next, wire no. 1 is moved in the $\bar{y}$ direction until its wake is sensed by a temperature rise in wire no. 2. The position of wire no. 2 with respect to wire no. 1 then determines the flow direction.

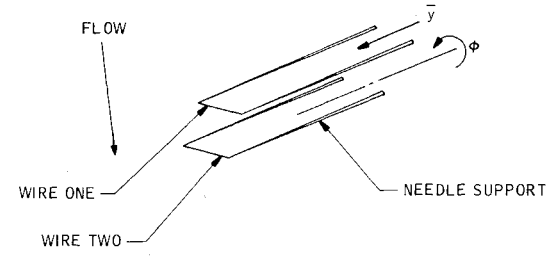

Fig. 1 Probe schematic diagram; a short current pulse heats wire no. 1, which is moved so that its heated wake is sensed by wire no. 2; the relative positions of the two wires and the time for the heat pulse to reach wire no. 2 determine the velocity vector.

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* Principal Scientist, Fluid Mechanics Department. Member AIAA.